# Adaptive Linear Regression Selection

Hung Chen

Department of Mathematics
Joint work with Mr. Chiuan-Fa Tang
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# Linear Regression Models

Consider a linear regression model with normal error,

$$\mathbf{Y} = \boldsymbol{\mu} + \boldsymbol{\epsilon} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon},$$

where

- $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_p)$ is an $n \times p$ matrix,
- $\boldsymbol{\beta} = (\beta_1 \ \ldots, \beta_p)^T$,
- $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)^T = \mathbf{X}\boldsymbol{\beta}$,
- $\boldsymbol{\epsilon} = (\varepsilon_1, \ldots, \varepsilon_n)^T \sim N(\mathbf{0}, \sigma^2\mathbf{I})$, and $\sigma^2$ is known.

# Nested Models

We only consider the nested linear competing model

$$\{M_k, k = 0, \ldots, p\}.$$

- Lasso leads to a data-driven nested models.
- For model $M_k$, $\beta_j \neq 0$ for $j \leq k$ and $\beta_j = 0$ for $j > k$.
- $\beta$'s are estimated by the **least square method** and
- $\boldsymbol{\mu}$ is estimated by

$$\hat{\boldsymbol{\mu}}_{M_k} = P_{M_k}\mathbf{Y},$$

where $P_{M_k}$ is the projection matrix corresponding to model $M_k$.

- Its residual sum of squares is defined as

$$RSS(M_k) = \left(\mathbf{Y} - \hat{\boldsymbol{\mu}}_{M_k}\right)^T \left(\mathbf{Y} - \hat{\boldsymbol{\mu}}_{M_k}\right).$$

# Model Selection

If AIC (Mallows' $C_p$) is used to score models, we choose the model $\hat{M}$ by minimizing

$$RSS(M_k) + 2|M_k|\sigma^2$$

with respect to all competing models $\{M_k, k = 0, \ldots, p\}$, where $|M_k|$ is the size of $M_k$.
Note that

- It does not include the random error introduced in model selection procedure.
- What can be done?
  - Refer to the proposal in Shen and Ye (2002).

# Shen and Ye's proposal (2002, *JASA*)

Shen and Ye (2002) proposed to choose $\lambda > 0$ to minimize the unbiased risk estimator

$$\hat{\lambda} = \mathit{argmin}_{\lambda>0} \quad \mathit{RSS}(\hat{M}(\lambda)) + g_0(\lambda)\sigma^2 \quad .$$

The resulting selected model is $\hat{M}(\hat{\lambda})$.
As an attempt to understand their proposal, consider the situation

- BIC is consistent (no underfitting).
- nested competing models
- $\lambda \in [0, \log n]$

Is

$$\hat{M}(\hat{\lambda}) = \hat{M}(\log n) = M_{k_0}$$

or $\hat{\lambda} = \log n$?

# Assumptions: BIC is consistent

Recall that $p_0$ is the number of covariates in the true model. Assume that

Assumption B1. There exists a constant $c > 0$ such that $\boldsymbol{\mu}^T(\mathbf{I} - \mathbf{P}_{M_k})\boldsymbol{\mu} \geq cn$ for all $k < p_0$, where

$$\boldsymbol{\mu} = \mathbf{X}_{p_0}(\beta_1, \ldots, \beta_{p_0})^T$$

is the mean vector of the true model.

Assumption B2. The simple size $n$ is large enough such that $cn > 2p_0 \log n$.

Assumption N. $\log n > 2\log(p - p_0)$.

# Determine $_0(\lambda)$.

It follows from the results of Spitzer (1956), Woodroofe (1982) and Zhang (1992) that, for all $\lambda \in [0, \log n]$,

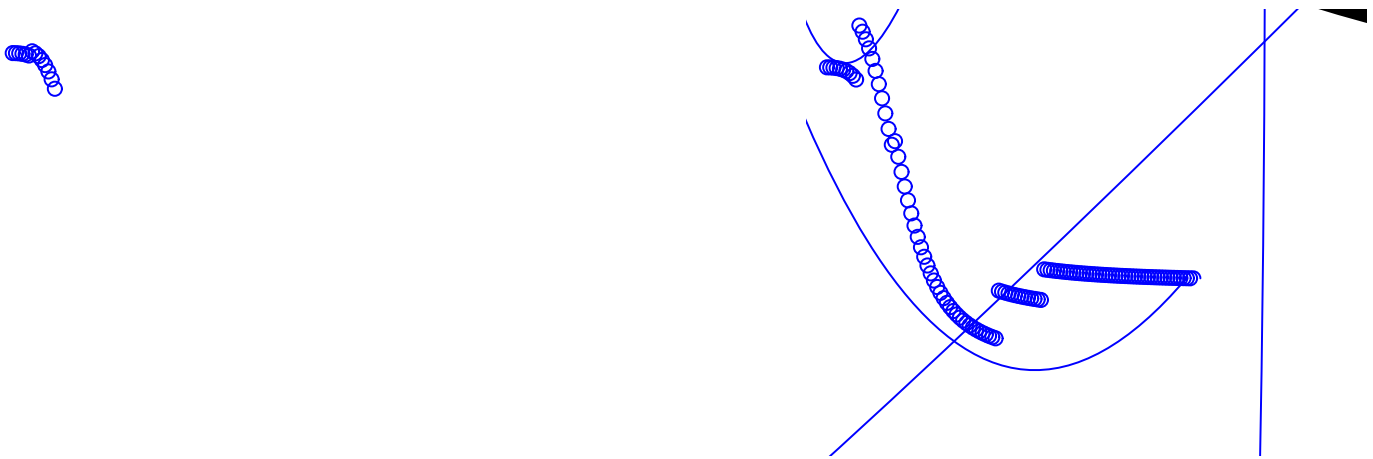
$$g_0(\lambda) = 2 \sum_{j=1}^{p-p_0} \left[ P(\chi^2_{j+2} > j\lambda) \right] + 2p_0.$$

Note that

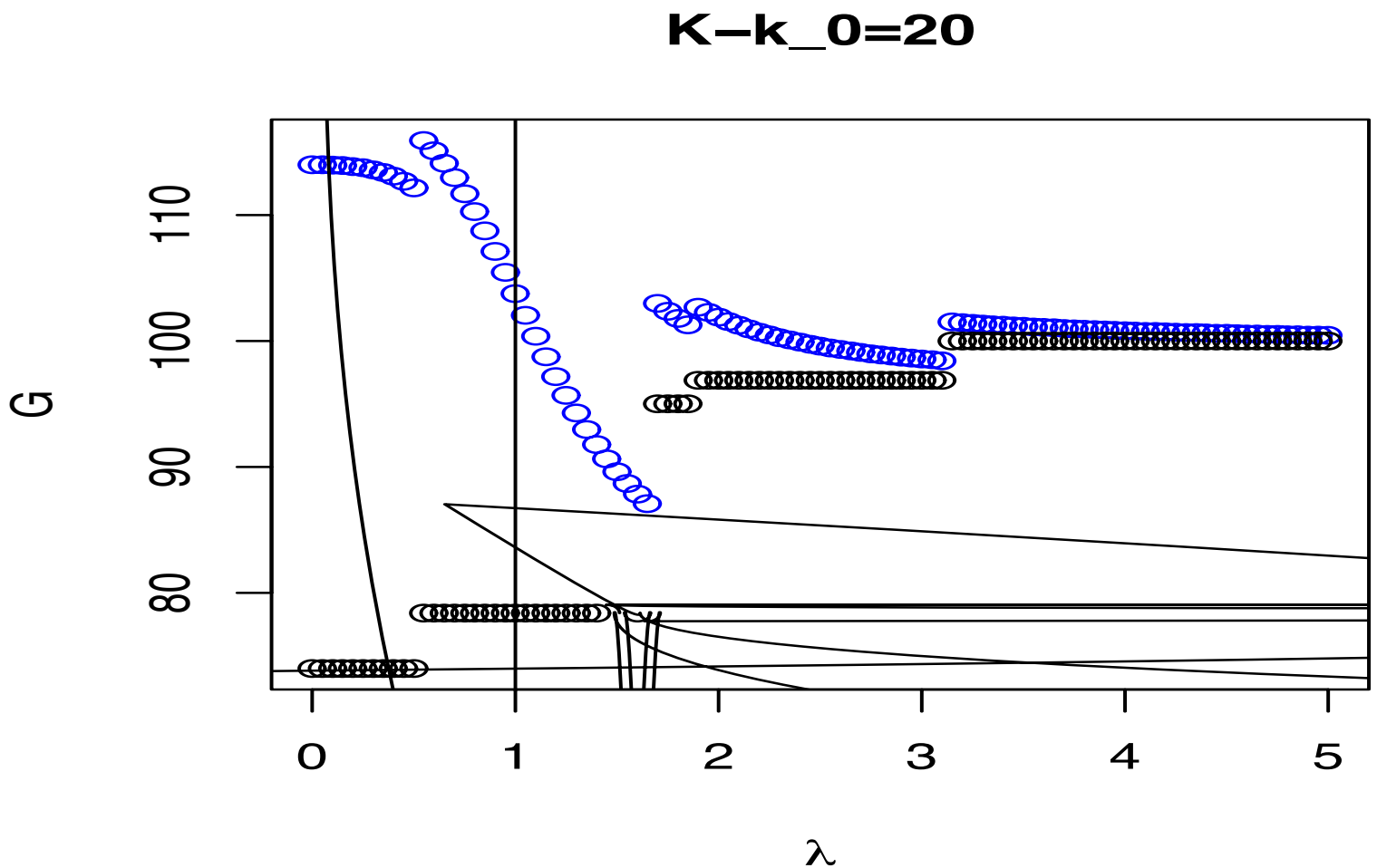
- $g_0(\lambda)$ is strictly decreasing.
- $g_0(0) = 2p$.
- $g_0(\log n) \to 2p_0$ as $n \to \infty$.

# AMS improves.

Consider a simulation study with $p_0 = 0$, $p - p_0 = 20$, $n = 404$ ($\log n = 6$), and $\sigma^2 = 1$.
The black points are $RSS(\hat{M}(\lambda)) - RSS(M_{p_0})$ and the blue points are $RSS(\hat{M}(\lambda)) + g_0(\lambda) - RSS(M_{p_0})$.



# *AMS* may not work but how often?



# Probability of correct selection:

| $\hat{M}(\hat{\lambda}) = M_{p_0+}$ | $[0, \log n]$ | $[0.5, \log n]$ | $[1, \log n]$ | $[1.5, \log n]$ | $[2, \log n]$ |
|---|---|---|---|---|---|
| 0 | 0.5457 | 0.5457 | 0.5457 | 0.6483 | 0.7539 |
| 1 | 0.0565 | 0.0565 | 0.0565 | 0.0681 | 0.0807 |
| 2 | 0.0312 | 0.0312 | 0.0312 | 0.0386 | 0.0474 |
| 3 | 0.0262 | 0.0262 | 0.0262 | 0.0320 | 0.0348 |
| 4 | 0.0239 | 0.0239 | 0.0239 | 0.0283 | 0.0249 |
| 5 | 0.0188 | 0.0188 | 0.0188 | 0.0227 | 0.0166 |
| 6 | 0.0156 | 0.0156 | 0.0156 | 0.0190 | 0.0103 |
| 7 | 0.0134 | 0.0134 | 0.0134 | 0.0169 | 0.0071 |
| 8 | 0.0136 | 0.0136 | 0.0136 | 0.0157 | 0.0051 |
| 9 | 0.0140 | 0.0140 | 0.0140 | 0.0151 | 0.0041 |
| 10 | 0.0155 | 0.0155 | 0.0155 | 0.0132 | 0.0039 |
| 11 | 0.0155 | 0.0155 | 0.0155 | 0.0107 | 0.0022 |
| 12 | 0.0153 | 0.0153 | 0.0153 | 0.0106 | 0.0018 |
| 13 | 0.0163 | 0.0163 | 0.0163 | 0.0097 | 0.0018 |
| 14 | 0.0177 | 0.0177 | 0.0177 | 0.0080 | 0.0015 |
| 15 | 0.0185 | 0.0185 | 0.0185 | 0.0074 | 0.0012 |
| 16 | 0.0210 | 0.0210 | 0.0210 | 0.0070 | 0.0008 |
| 17 | 0.0242 | 0.0242 | 0.0242 | 0.0074 | 0.0005 |
| 18 | 0.0212 | 0.0212 | 0.0212 | 0.0069 | 0.0006 |
| 19 | 0.0307 | 0.0307 | 0.0307 | 0.0065 | 0.0005 |
| 20 | 0.0452 | 0.0452 | 0.0452 | 0.0079 | 0.0003 |

# Need a detailed description of $_0(\lambda)$

Recall

$$\hat{\lambda} = \min_{\lambda>0}\{\lambda : RSS(\hat{M}(\lambda)) + g_0(\lambda)\}$$

and choose model $\hat{M}(\hat{\lambda})$ which retains the first $\hat{j}(\hat{\lambda})$ predictors.

- When $\lambda = 0$, $|\hat{M}(0)| = p$ for all realizations and $RSS(\hat{M}(0)) = \mathbf{Y}^T(\mathbf{I} - \mathbf{P}_p)\mathbf{Y}$. Then $g_0(0) = 2p$.
- When $\lambda = \ln n$, $|\hat{M}(\ln n)| = p_0$ for almost all realizations and $RSS(\hat{M}(\ln n)) = \mathbf{Y}^T(\mathbf{I} - \mathbf{P}_{p_0})\mathbf{Y}$. Then $g_0(\ln n) = 2p_0$.

Note that

$$RSS(\hat{M}(0)) + 2p\hat{\sigma}^2 - RSS(\hat{M}(\ln n)) + 2p_0\hat{\sigma}^2 = \sigma^2 \sum_{k=1}^{p-p_0} (2 - V_k)$$

which is greater than 0 with probability close to 1 when $p - p_0$ is large.

# Adaptive selection over $\lambda \in [0, 0.5] \cup \{\log \ \}$

Show that $\hat{\lambda} = \log n$ with probability close to 1 by finding a bound on the following probability.

$$P\left(RSS(\hat{j}(\lambda)) + g_0(\lambda) < RSS(\hat{j}(\ln n)) + g_0(\ln n) \ \text{ for all } \lambda \in [0, 0.5]\right).$$

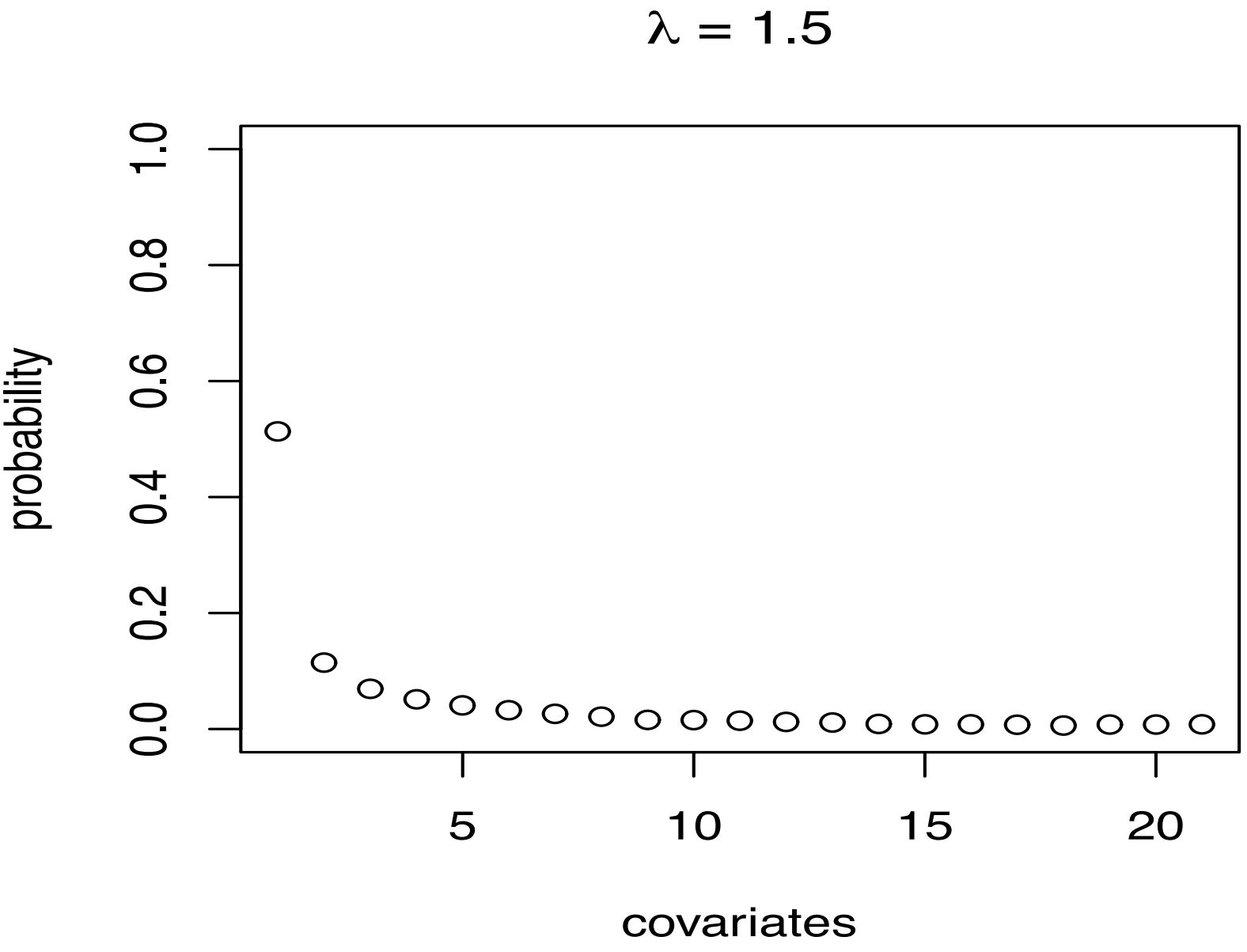
Note that

$$\begin{aligned} & P\left(V_1 + \cdots + V_{\hat{j}(\lambda)} < g_0(\lambda) \ \text{ for all } \lambda \in [0.0.5]\right) \\ \geq \ & P\left(V_1 + \cdots + V_{p-p_0} < g_0(0) - 4\right) \\ = \ & P\left(V_1 + \cdots + V_{p-p_0} < 2(p - p_0) - 4\right). \end{aligned}$$

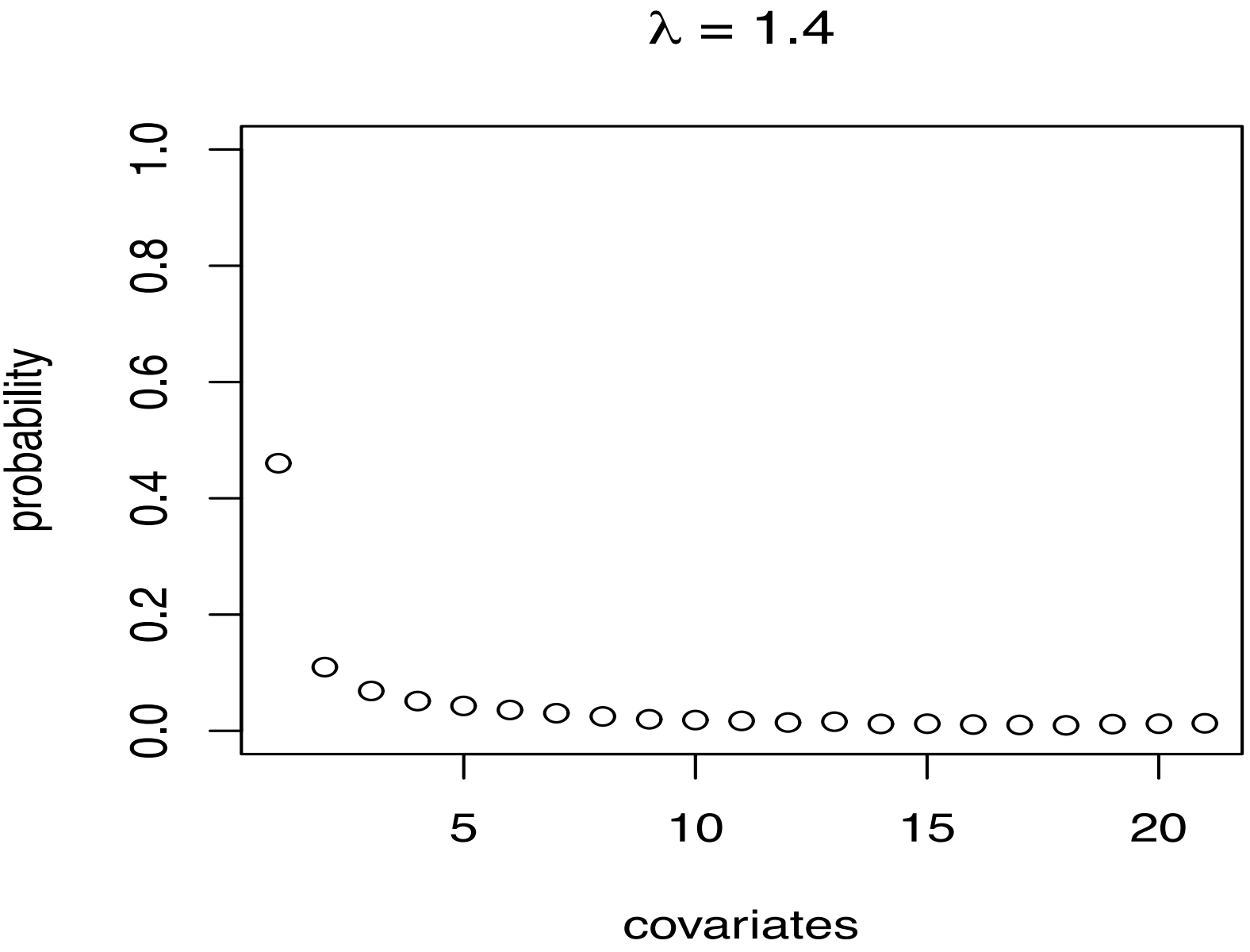
Note that

- $g_0(\lambda)$ is strickly decreasing and continuous on $\lambda \in [0, \ln n]$.
- For all $g_0(\ln n) < \delta \leq g_0(0)$, there exists a unique $\lambda_\delta$ such that $g_0(\lambda_\delta) = g_0(0) - \delta$.
- Claim: When $\delta = 4$, $0.5 \leq \lambda_\delta$.

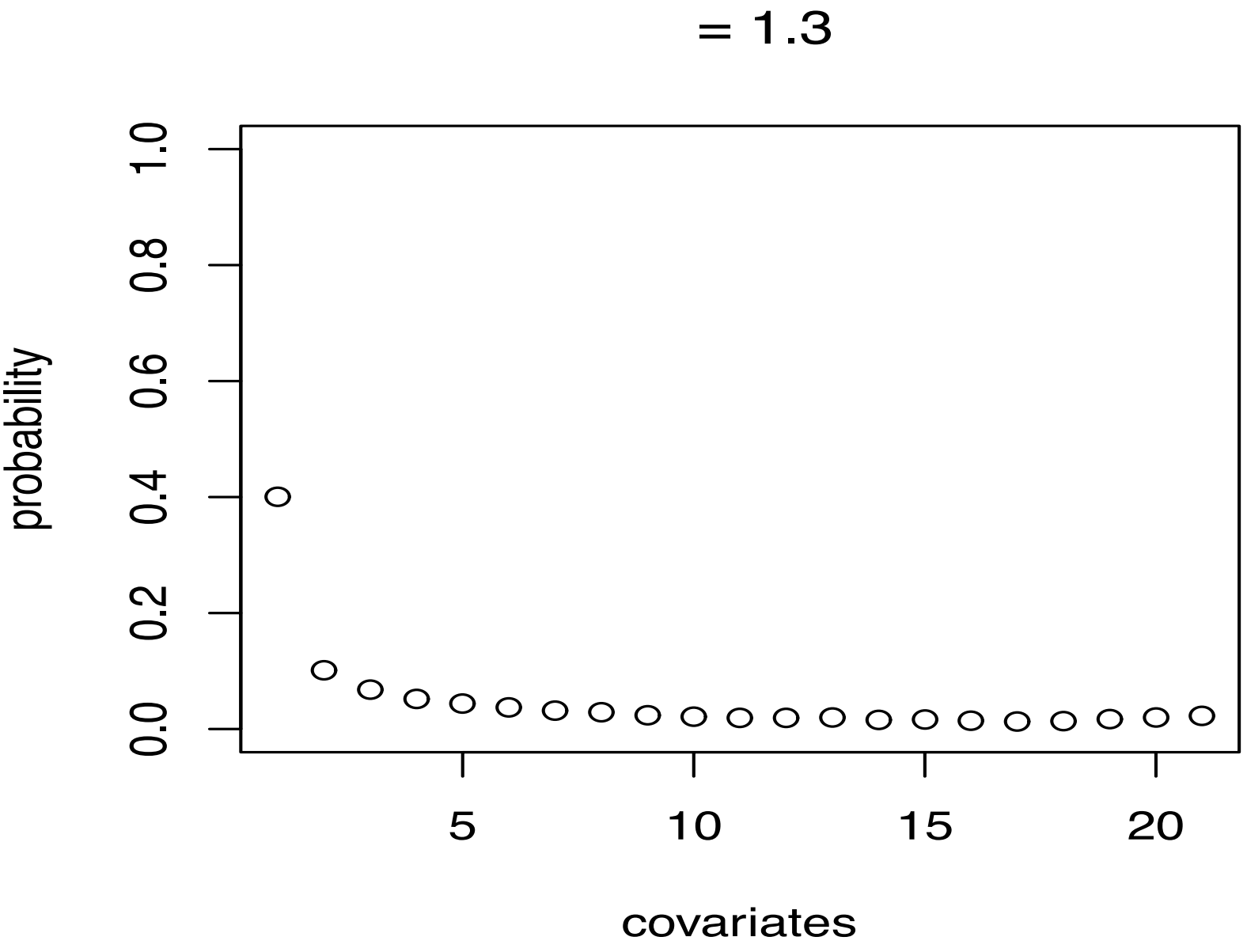
# Simulation of $\{S_k(1.5)\}$



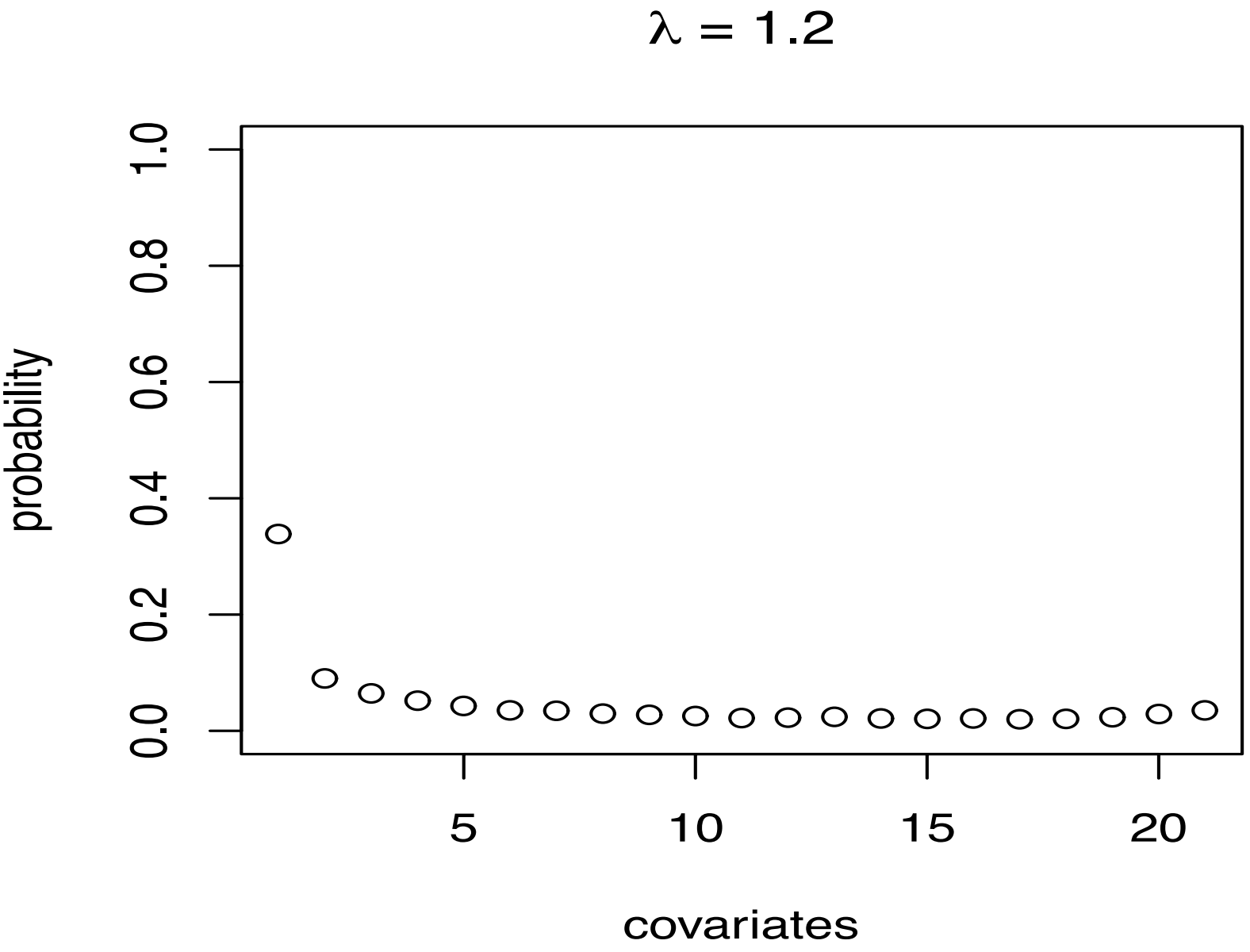
# Simulation of $\{S_k(1.4)\}$



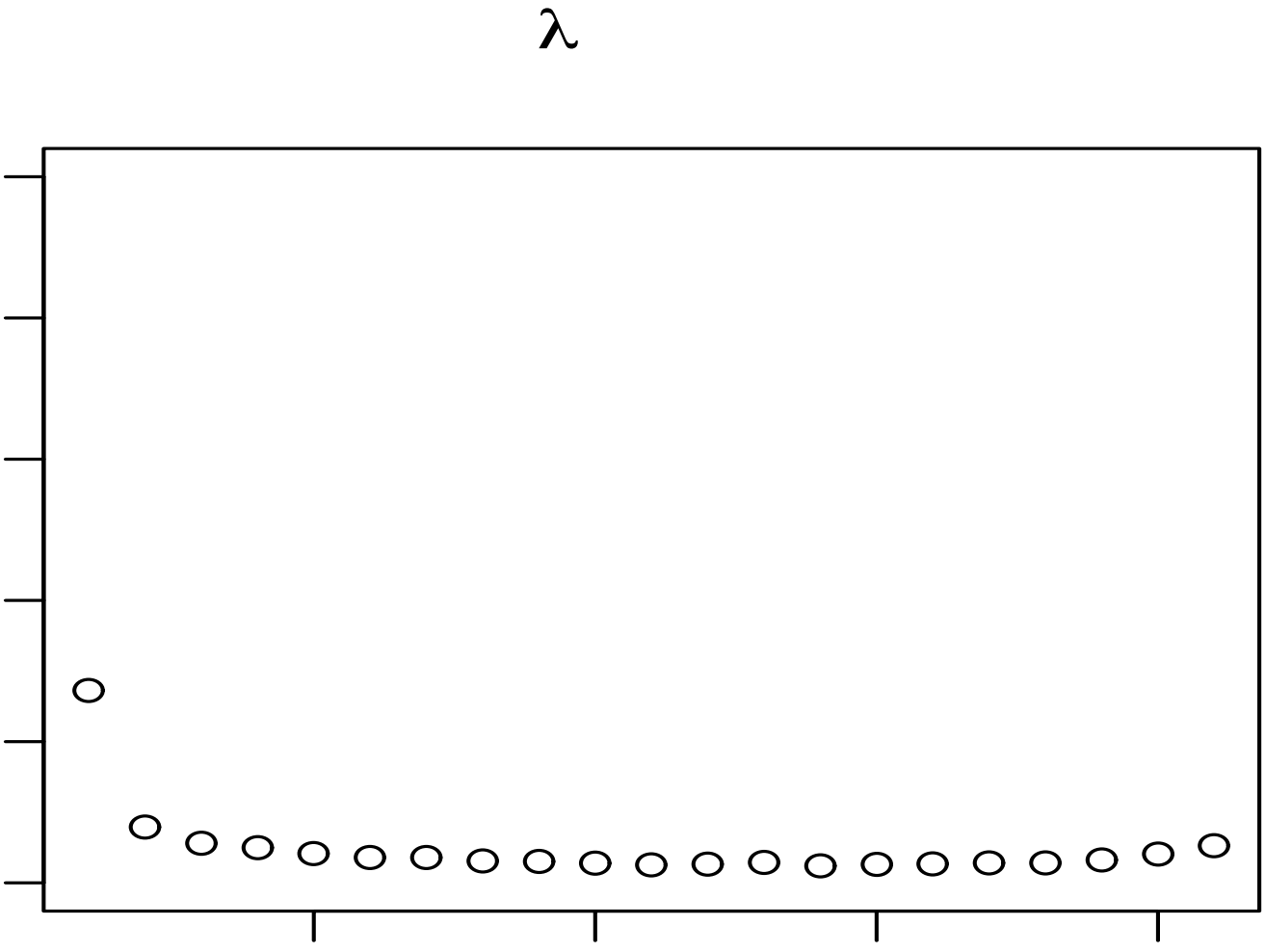
# Simulation of $\{S_k(1.3)\}$



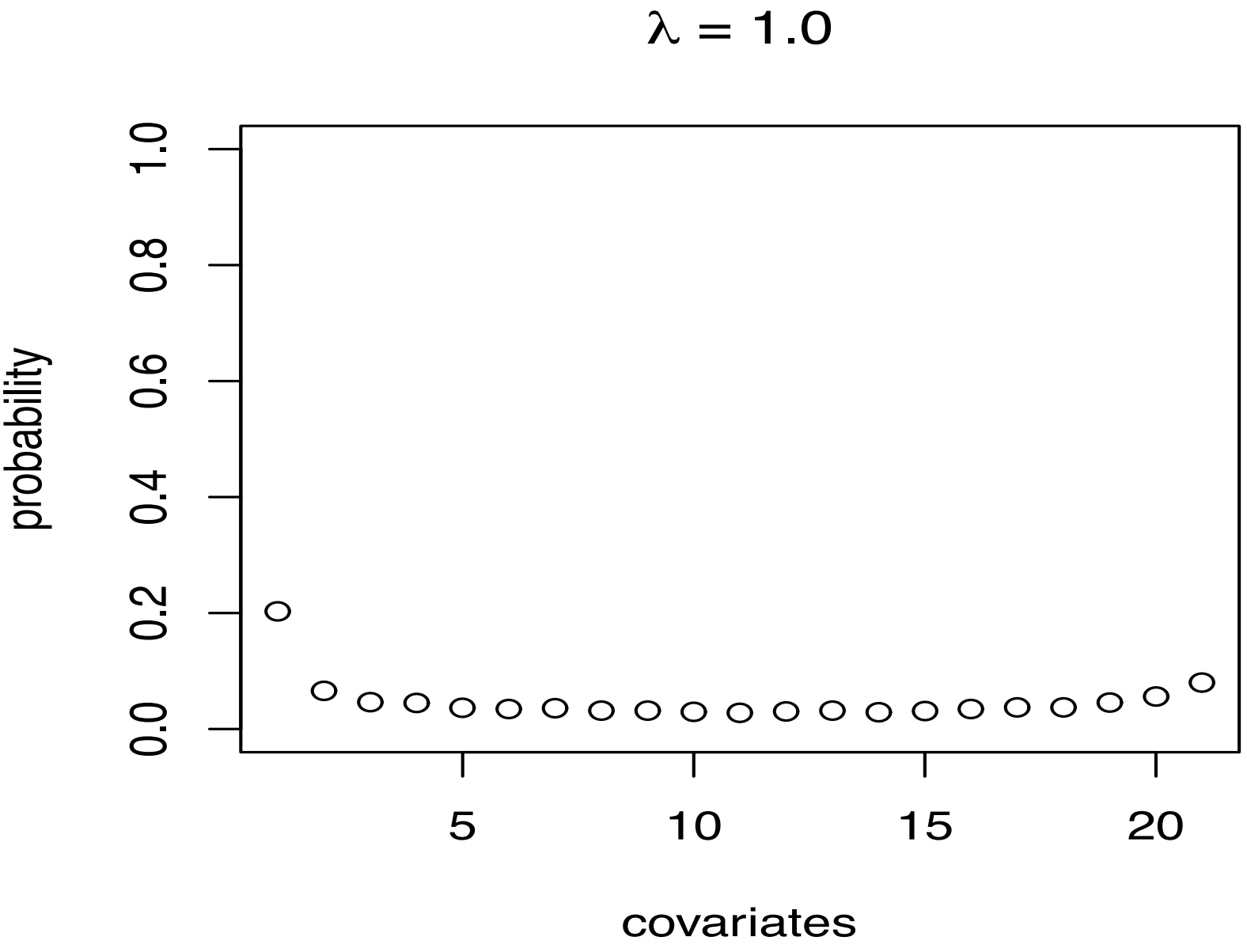
# Simulation of $\{S_k(1.2)\}$



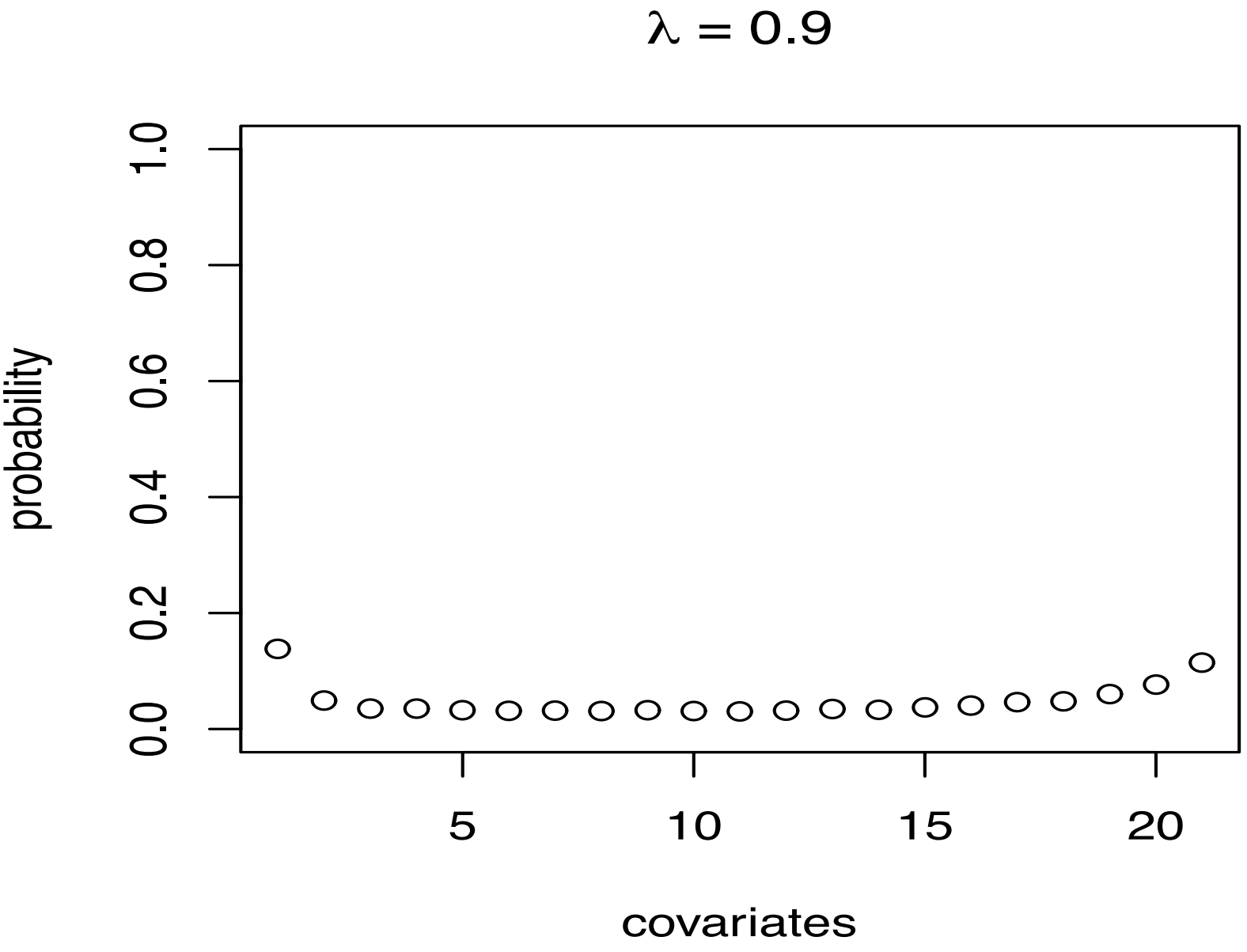
# Simulation of $\{S_k(1.1)\}$



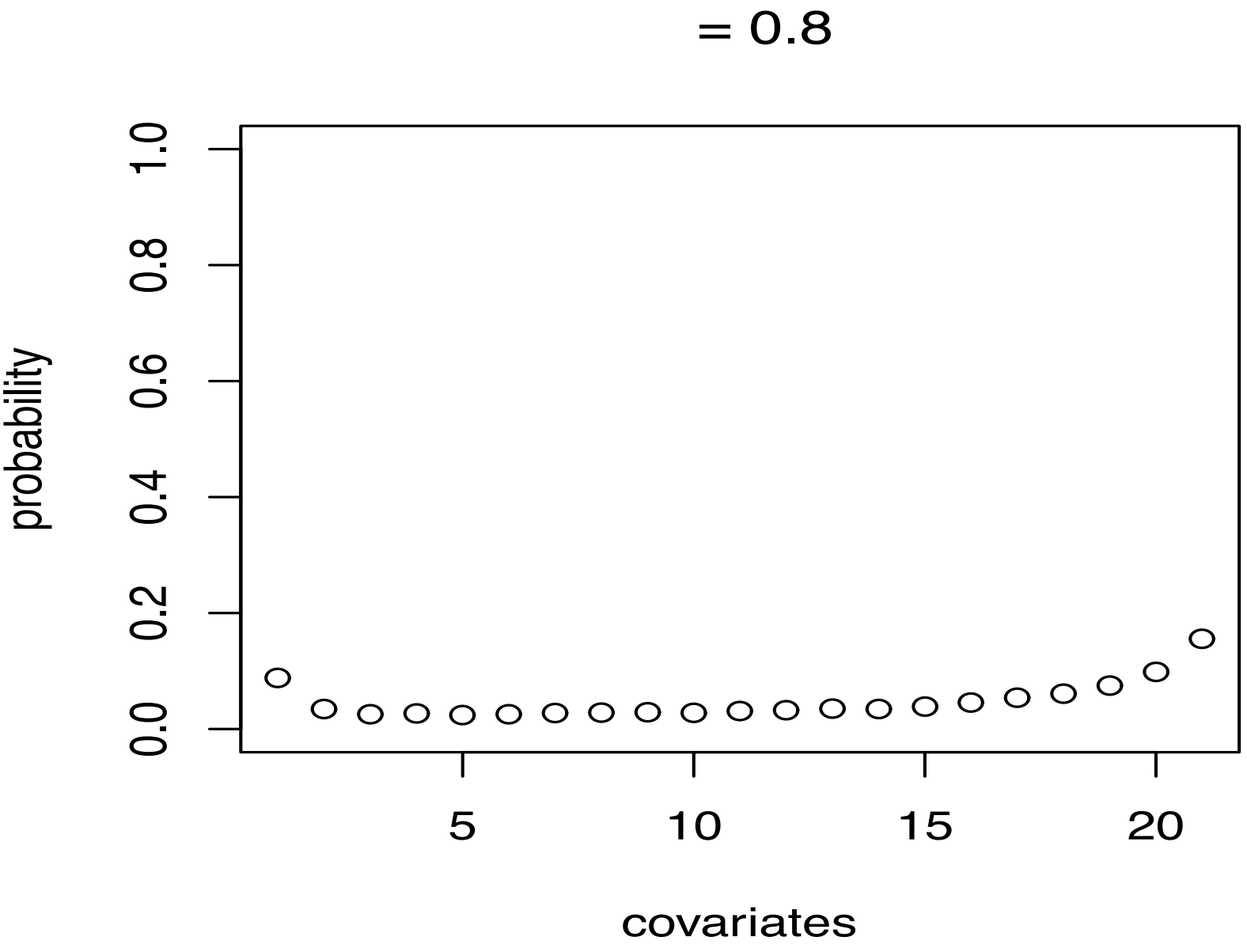
# Simulation of $\{S_k(1.0)\}$



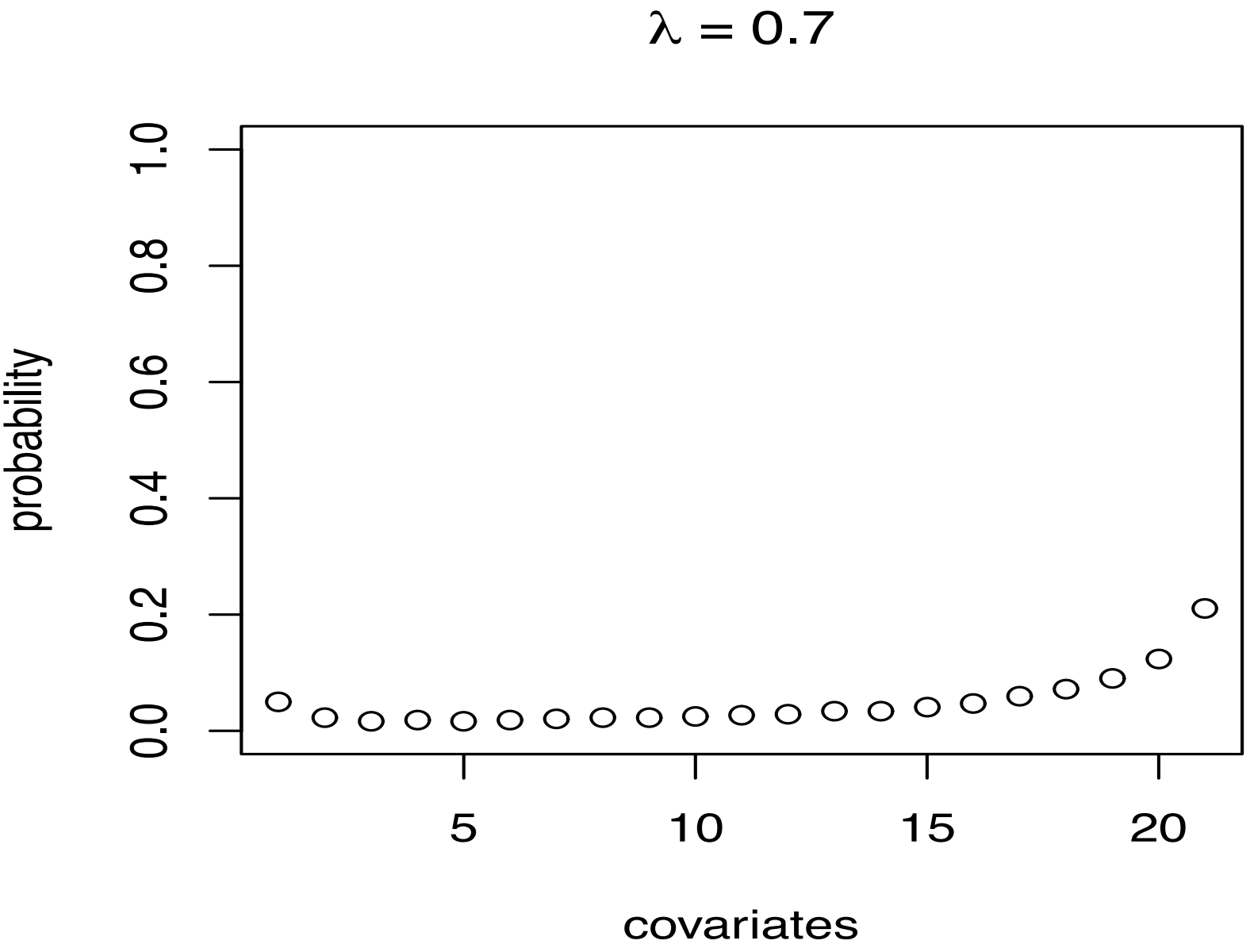
# Simulation of $\{S_k(0.9)\}$



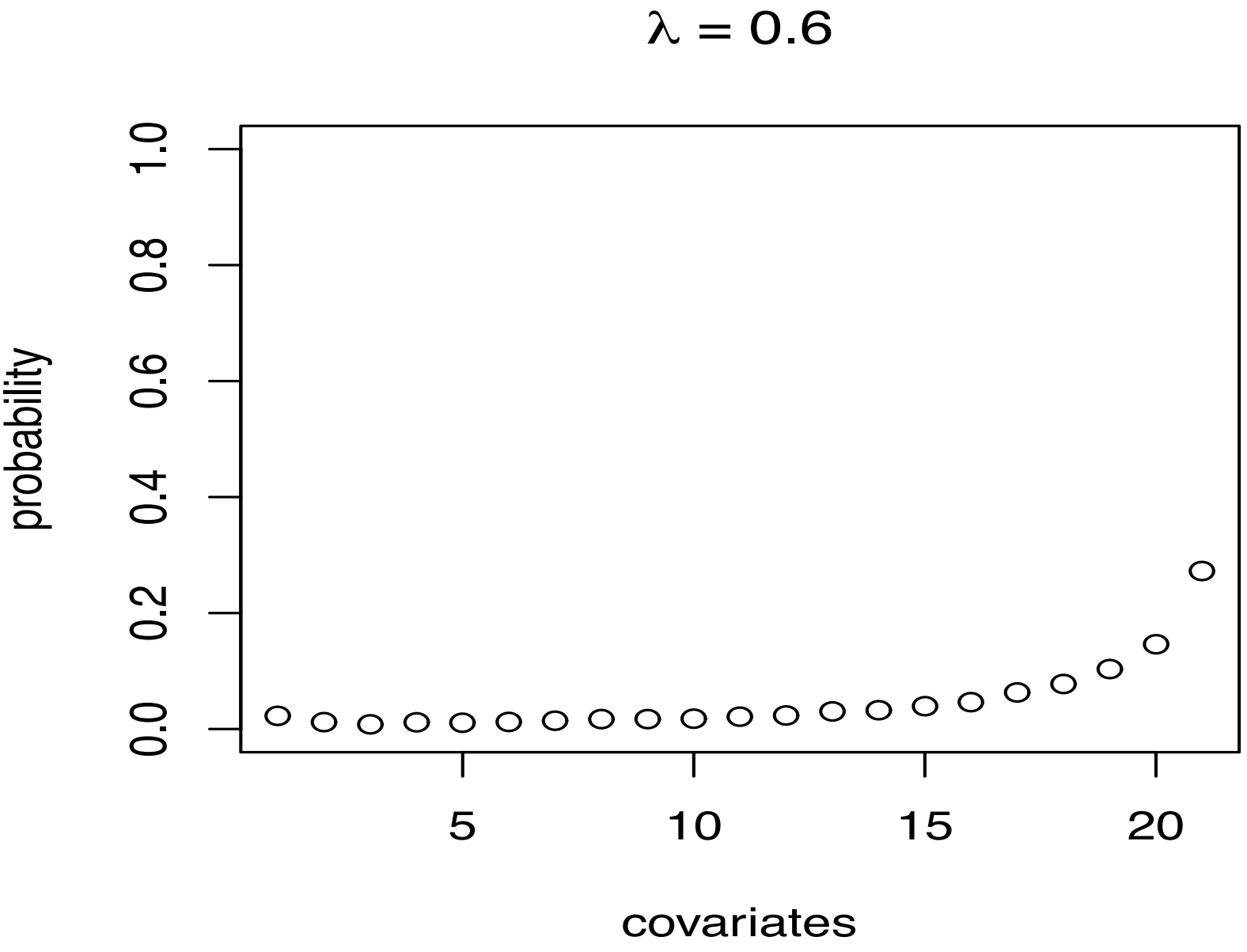
# Simulation of $\{S_k(0.8)\}$



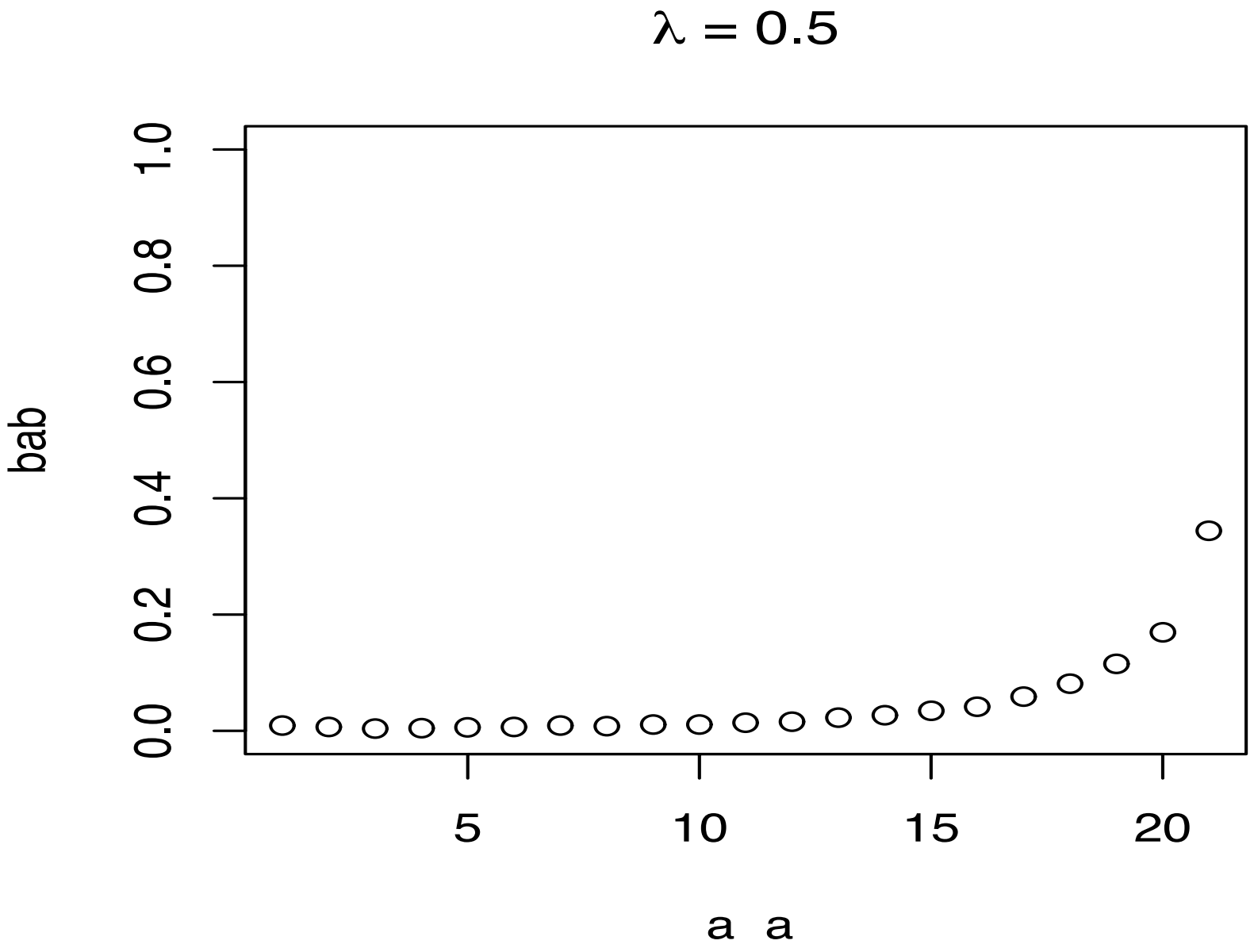
# Simulation of $\{S_k(0.7)\}$



# Simulation of $\{S_k(0.6)\}$



# Simulation of $\{S_k(0.5)\}$



# Conclusion

- When $\lambda \in (2, \log n]$, there are about 75% to choose the true model.
- The probability of selecting correct model decreases to 55% if $\lambda \in [1, 2) \cup [2, \log n]$.
- For the region of $\lambda$ are $[0, \log n]$, $\in [0.5, \log n]$, or $n[1, \log n]$, there are no differences in the probability of correct selection.
  - We still cannot provide a good interpretation.